Particle Physics Midterm Exam   Name ___KEY___

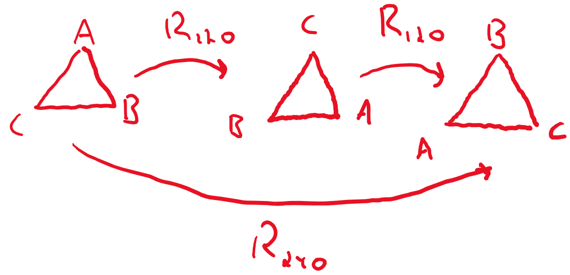
Your grade will be based on the six problems that you do best on. If on any problem you find yourself doing a ridiculous amount of tedious work, you are probably doing something wrong! If needed please use the metric convention that $\eta_{\mu\nu} = \mathrm{diag}(-1,1,1,1)$.

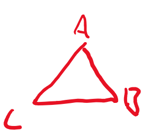
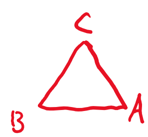
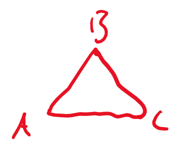
1. Consider the group of transformations in 2D which carry the corners of an equilateral triangle into corners.
   a) Construct the multiplication table for the elements of this group.
   b) Construct a faithful linear (matrix) representation of this group, showing the matrix form of all elements of the group. **Hint:** It helps to identify the "basic" transformations from which you can get everything else by repeated application.

a)

	I	R_{120}	R_{240}
I	I	R_{120}	R_{240}
R_{120}	R_{120}	R_{240}	I
R_{240}	R_{240}	I	R_{120}

since



b) Call:  $= \begin{pmatrix}1\\0\\0\end{pmatrix}$  $= \begin{pmatrix}0\\1\\0\end{pmatrix}$  $= \begin{pmatrix}0\\0\\1\end{pmatrix}$

Then: $I = \begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix}$ $R_{120} = \begin{pmatrix}0&0&1\\1&0&0\\0&1&0\end{pmatrix}$ $R_{240} = R_{120}^2 = \begin{pmatrix}0&1&0\\0&0&1\\1&0&0\end{pmatrix}$

Alternatively you could use 2D:

$I = \begin{pmatrix}1&0\\0&1\end{pmatrix}$ $R_{120} = \begin{pmatrix}\cos 120 & \sin 120\\ -\sin 120 & \cos 120\end{pmatrix} = \begin{pmatrix}-\frac{1}{2} & \frac{\sqrt{3}}{2}\\ -\frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$ $R_{240} = R_{120}^2 = \begin{pmatrix}-\frac{1}{2} & -\frac{\sqrt{3}}{2}\\ \frac{\sqrt{3}}{2} & -\frac{1}{2}\end{pmatrix}$

Which acts on: $\begin{pmatrix}0\\1\end{pmatrix}, \begin{pmatrix}\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{pmatrix}, \begin{pmatrix}-\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{pmatrix}$

Better still you can use 1D:

$I = 1$ $R_{120} = -\frac{1}{2} + i\frac{\sqrt{3}}{2}$ $R_{240} = -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

Which acts on: $1, -\frac{1}{2} + i\frac{\sqrt{3}}{2}, -\frac{1}{2} - i\frac{\sqrt{3}}{2}$

2. Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ and the transformation for $SO(1,1)$, i.e. $\Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \gamma & -\beta\gamma \\ -\beta\gamma & \gamma \end{pmatrix}$, with $\beta = \frac{1}{2}$. Note that with this you can also find the value for γ.
Write down the transformed components of M if it is a $(1,1)$ tensor.

First of all if $\beta = \frac{1}{2}$ then $\gamma = \frac{1}{\sqrt{1-\beta^2}} = \frac{2}{\sqrt{3}} \Rightarrow \Lambda^{\mu'}{}_{\mu} = \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix} \equiv \Lambda$

Then: $M_\mu{}^\nu \rightarrow M_{\mu'}{}^{\nu'} = \Lambda^{\mu}{}_{\mu'} \Lambda^{\nu'}{}_{\nu} M_\mu{}^\nu$ and $\Lambda^T = \Lambda$ while

$\Lambda^{-1} = \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix} = \Lambda^{-1\,T}$

$$= \Lambda^{\mu}{}_{\mu'} M_\mu{}^\nu \Lambda^{\nu'}{}_{\nu}$$

$$= \Lambda^{-1\,T} M \Lambda^T$$

$$= \begin{pmatrix} \frac{2}{\sqrt{3}} & \frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} \frac{2}{\sqrt{3}} & -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} & \frac{2}{\sqrt{3}} \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$

Alternatively, if $M^\mu{}_\nu$ then:

$$M^\mu{}_\nu \rightarrow M^{\mu'}{}_{\nu'} = \Lambda^{\mu'}{}_{\mu} \Lambda^{\nu}{}_{\nu'} M^\mu{}_\nu = \Lambda^{\mu'}{}_{\mu} M^\mu{}_\nu \Lambda^{\nu}{}_{\nu'}$$

$$= \Lambda M \Lambda^{-1}$$

$$= \frac{1}{3} \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix}$$

$$= \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$$ same answer!

3. The generators of SU(3) can be written as $g_i = \frac{\lambda_i}{2}$ where: $\lambda_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_2 =$
$\begin{pmatrix} 0 & -i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \lambda_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},$

$$\lambda_5 = \begin{pmatrix} 0 & 0 & -i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \lambda_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \lambda_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -i \\ 0 & i & 0 \end{pmatrix}, \lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -2 \end{pmatrix}$$

where the nonzero associated structure constants of the Lie Algebra are

$$f^{123} = 1, \quad f^{147} = f^{165} = f^{246} = f^{257} = f^{345} = f^{376} = \frac{1}{2}, \quad f^{458} = f^{678} = \frac{\sqrt{3}}{2}$$

with f^{ijk} being totally antisymmetric in the three indices, i.e. $f^{ijk} = -f^{jik}$.

a) Do any pair of these generators commute? If so, identify at least one pair. If not argue why.

b) Find the commutator of g_6 and g_7.

a) Any pair ij for which f^{ijk} (or any permutation) is zero will commute. So $[g_1, g_8] = 0$, $[g_2, g_8] = 0$ and $[g_3, g_8] = 0$.

b) Two ways:

$$[g_6, g_7] = \frac{1}{4}\left[\begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix} - \begin{pmatrix} 0 & & \\ & 0 & -i \\ & i & 0 \end{pmatrix}\begin{pmatrix} 0 & & \\ & 0 & 1 \\ & 1 & 0 \end{pmatrix}\right] = \frac{1}{4}\left[\begin{pmatrix} 0 & & \\ & i & \\ & & -i \end{pmatrix} - \begin{pmatrix} 0 & & \\ & -i & \\ & & i \end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

or:

$$[g_6, g_7] = i f^{67k} g_k = i f^{678} g_8 + i f^{673} g_3 = i\left[\frac{\sqrt{3}}{2} g_8 - \frac{1}{2} g_3\right]$$

$$= \frac{i}{2}\left[\frac{\sqrt{3}}{2\sqrt{3}}\begin{pmatrix} 1 & & \\ & 1 & \\ & & -2 \end{pmatrix} - \frac{1}{2}\begin{pmatrix} 1 & & \\ & -1 & \\ & & 0 \end{pmatrix}\right]$$

$$= \frac{i}{2}\begin{pmatrix} 0 & & \\ & 1 & \\ & & -1 \end{pmatrix}$$

4. Recall that for 4-vectors the dot product is $P^1 \cdot P^2 \equiv P^1_\mu P^{2\mu}$.
Evaluate $Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)P^1_\mu P^2_\nu P^2_\lambda P^1_\rho$ in terms of 4-vector dot products, e.g. $P^1 \cdot P^2$, etc.

From HW 4 we know:

$$Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho) = 4(\eta^{\mu\nu}\eta^{\lambda\rho} - \eta^{\mu\lambda}\eta^{\nu\rho} + \eta^{\mu\rho}\eta^{\nu\lambda})$$

So:

$$\begin{aligned} Tr(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\rho)P^1_\mu P^2_\nu P^2_\lambda P^1_\rho = 4\big[&\eta^{\mu\nu}\eta^{\lambda\rho} P^1_\mu P^2_\nu P^2_\lambda P^1_\rho \\ &- \eta^{\mu\lambda}\eta^{\nu\rho} P^1_\mu P^2_\nu P^2_\lambda P^1_\rho \\ &+ \eta^{\mu\rho}\eta^{\nu\lambda} P^1_\mu P^2_\nu P^2_\lambda P^1_\rho\big] \\ = 4\big[&P^1 \cdot P^2 \; P^2 \cdot P^1 \\ &- P^1 \cdot P^2 \; P^2 \cdot P^1 \\ &+ P^1 \cdot P^1 \; P^2 \cdot P^2\big] \\ = 4 &P^{1^2} P^{2^2} \end{aligned}$$

5. Consider the projection operators $P_{\pm} = \frac{1}{2}\left(1 \pm \frac{2}{\hbar} S_{\vec{p}}\right)$.

Show that $P_+ P_- \psi^{(1)} = 0$. You may do this either in terms of an explicit example or by manipulation of the operator definition.

$$P_+ P_- \psi^{(1)} = \frac{1}{4}\left(1 + \frac{2}{\hbar} S_{\vec{p}} - \frac{2}{\hbar} S_{\vec{p}} - \frac{4}{\hbar^2} S_{\vec{p}}^2\right)\psi^{(1)}$$

$$= \frac{1}{4}\left(1 - \frac{4}{\hbar^2}\frac{\hbar^2}{4}\right)\psi^{(1)}$$

$$= 0$$

6. Construct a gauge theory for scalar fields invariant under a local $SO(2)$. For the following you do **not** need to show your work, but you must be every clear in notating your answers.

a) Write down a starting Lagrangian for this theory, specifying the form of the global transformation on the matter fields.

b) Promote this to a local symmetry by writing down a covariant derivative and transformation rule for the gauge field(s).

c) Write down gauge invariant kinetic term(s) for the gauge field(s). You must include the explicit form of $F_{\mu\nu}$.

d) Beer.

a) $\mathcal{L} = \frac{1}{2}\partial_\mu \phi^T \partial^\mu \phi + \left(\frac{mc}{\hbar}\right)^2 \phi^T \phi$

$\phi = \begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix} \to \phi' = U\phi \Rightarrow \phi^T \to \phi'^T = (U\phi)^T = \phi^T U^T$

$U \in SO(2)$ which is abelian so everything commutes

b) $\partial_\mu \to D_\mu = \partial_\mu + A_\mu$

We want: $D_\mu \phi \to D'_\mu \phi' = U D_\mu \phi$

So we need: $D'_\mu \phi' = (\partial_\mu + A'_\mu) U\phi$

$= (\partial_\mu U)\phi + U\partial_\mu \phi + A'_\mu U \phi$

Which we want to be: $= U(\partial_\mu \phi + A_\mu \phi) \Rightarrow A'_\mu = A_\mu - (\partial_\mu U)U^{-1}$

c) Then add $-\frac{1}{16\pi} F_{\mu\nu}F^{\mu\nu}$ w/ $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

d) 

7. Consider a Lagrangian for four real scalar fields $(\varphi_1, \varphi_2, \varphi_3, \varphi_4)$ of the form:

$$\mathcal{L} = \frac{1}{2}\partial_\mu\varphi_1\partial^\mu\varphi_1 + \frac{1}{2}\partial_\mu\varphi_2\partial^\mu\varphi_2 + \frac{1}{2}\partial_\mu\varphi_3\partial^\mu\varphi_3 + \frac{1}{2}\partial_\mu\varphi_4\partial^\mu\varphi_4 - \frac{\mu^2}{2}(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + +\varphi_4^2) + \frac{\lambda^2}{4}(\varphi_1^2 + \varphi_2^2 + \varphi_3^2 + \varphi_4^2)^2$$

If this four component scalar field is the Higgs field, determine the number and mass of the Higgs boson(s) as well as the number of Goldstone boson(s).

This potential is hyper-spherically symmetric, so we can't draw it, but nonetheless given the spherical nature, we identify the Higgs boson as the single radial fluctuation w/ mass $m_H = \frac{\sqrt{2}\hbar\mu}{c}$.

Then, since there are three dimensions orthogonal to the radial direction (surface of S^3), we expect 3 Goldstone bosons.